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Multiplicative Transport Processes in One Dimension

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1. INTRODUCTION

The following model for the multiplication of neutrons in a one-dimensional rod has been studied by Bellman, Kalaba, and Wing [1, 2, 3] in order to clarify the concepts of criticality and critical size. It is assumed that neutrons travel with constant direction between collisions, the probability of a neutron making a collision in any small interval δs of path-length being $\lambda\delta s + o(\delta s)$. As a result of a collision the "parent" neutron is replaced by two neutrons which leave the point of collision in opposite directions and continue to move and multiply in the same manner as the parent neutron.

Bellman, Kalaba, and Wing use the principle of invariant imbedding (in which the process is considered as a function of the length of the rod) to determine the expected number of transmitted and reflected neutrons generated over all time by a single neutron incident at one end of the rod; they also consider the case of a particle which originates in the interior of the rod. In this way they are able to define a "critical length" for the rod as the smallest length for which the expected number of reflected neutrons becomes infinite. This is found to be $\pi/2\lambda$. They also derive equations for the probabilities

$$p_n(t) = \text{Prob [a single neutron incident on a rod of length } t \text{ eventually gives rise to a total of } n \text{ reflected neutrons]} \tag{1.1}$$

and for the corresponding generating function,

$$u(t, z) = \sum_{n=0}^{\infty} p_n(t)\, z^n, \tag{1.2}$$

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and make the following conjectures regarding the behavior of the probabilities $p_n(t)$:

$$\text{(a)} \quad p_n(t) < p_{n-1}(t), \qquad n = 1, 2, \cdots; t > 0, \tag{1.3}$$

(b) $p_n(t)$ is unimodal and has a maximum at $t = y_n$, where

$$\frac{\pi}{2\lambda} > y_n > y_{n-1} > 0$$

and

$$\lim_{n\to\infty} y_n = \frac{\pi}{2\lambda} = \text{critical length.} \tag{1.4}$$

In this paper we shall consider a modified model for neutron multiplication in which a random number, ν, of neutron pairs emerges from each collision (ν neutrons moving in each direction) in addition to the parent neutron which continues to move with its original direction of motion. Explicit solutions are derived for the probabilities $p_n(t)$ and it is shown in particular that there is a certain rod-length, t_c, such that

$$\sum_{n=0}^{\infty} p_n(t) < 1 \qquad \text{if} \qquad t > t_c\,.$$

This length, t_c, is also the smallest length for which the mean number of reflected particles becomes infinite; i.e., it is the "critical length" defined above. However the property that $\sum_{n=0}^{\infty} p_n(t) < 1$ for lengths greater than t_c clearly implies that the mean number of reflected particles is infinite, while the converse is not true. After a detailed analysis of the modified model, we shall return to the model of Bellman, Kalaba, and Wing and show that it also exhibits the property of criticality in this stronger sense.

In the particular case when the random variable ν is equal to 1 with probability 1, i.e., exactly two additional particles are produced at each collision, the conjectures (1.3) and (1.4) of Bellman, Kalaba, and Wing are established.

Finally a process is examined in which a random number of particles emerges from each collision, each particle moving to the left or right with probability $\frac{1}{2}$ independently of the others. A particular case of this model in which exactly two particles emerge from a collision has been studied by Harris [4].

The functions $\sum_{n=0}^{\infty} p_n(t)$ for the models of Harris [4] and Bellman, Kalaba, and Wing [1] have been investigated previously by T. W. Mullikin and R. N. Snow [9] using arguments of a different kind. In particular they derived for the latter model the function illustrated in Fig. 1, Section 5.

2. The Distribution of the Number of Emergent Particles

Consider a particle which originates at a point x in the rod $-a \leqslant x \leqslant a$. Its initial direction of motion will be denoted by a scalar μ ($\mu = 1$ for motion to the right and -1 for motion to the left). It moves in the manner described in the introduction; i.e., the probability of a collision in a small element of path-length δs is $\lambda \delta s + o(\delta s)$ and at each collision it gives rise to ν new particles (in addition to itself) moving in the same direction and ν new particles moving in the opposite direction; ν is a random variable taking the values $1, 2, \cdots$, with associated probabilities $\pi_1, \pi_2, \cdots$, respectively. Without loss of generality we shall assume throughout that the collision rate λ is equal to 1.

We introduce the probabilities $P_n(\mu, x)$ that a single particle originating at x with direction μ ultimately gives rise to a total of n particles emerging from the right-hand end of the rod. The corresponding generating function is defined by

$$G(\mu, x) = \sum_{n=0}^{\infty} P_n(\mu, x)\, z^n, \qquad |z| \leqslant 1. \tag{2.1}$$

Consideration of the point at which the first collision occurs gives the following integral equation for $G(\mu, x)$:

$$G(\mu, x) = z^{\delta_{1,\mu}} e^{(\mu x - a)} + \mu \int_x^{\mu a} G(\mu, y) \sum_{n=1}^{\infty} \pi_n G(\mu, y)^n\, G(-\mu, y)^n\, e^{\mu(x-y)}\, dy. \tag{2.2}$$

A derivation of this integral equation, for the more general case in which at each collision there are probabilities q_{ij} that the parent particle produces i particles moving in the same direction and j in the opposite direction, is given by Moyal [5]. Differentiating equation (2.2) with respect to x we find the backward Kolmogorov equation

$$\frac{d}{dx} G(\mu, x) = \mu G(\mu, x) \left[1 - \sum_{n=1}^{\infty} \pi_n G(\mu, x)^n\, G(-\mu, x)^n\right] \tag{2.3}$$

with boundary conditions

$$G(\mu, \mu a) = z^{\delta_{1,\mu}}. \tag{2.4}$$

From Eq. (2.3) it follows that $G(1, x)\, G(-1, x)$ is independent of x, i.e.,

$$G(1, x)\, G(-1, x) = K(z), \tag{2.5}$$

where K is a function of z with the properties

$$\begin{aligned} |K(z)| &\leqslant 1, \qquad |z| \leqslant 1, \\ |K(z)| &< 1, \qquad |z| < 1. \end{aligned} \tag{2.6}$$

Substituting $K(z)$ for $G(1, x)\, G(-1, x)$ in Eq. (2.3) and using the boundary conditions (2.4) we find that

$$G(\mu, x) = z^{\delta_{1,\mu}} \exp\left[(\mu x - a)\left(1 - \sum_{n=1}^{\infty} \pi_n K(z)^n\right)\right], \tag{2.7}$$

where $K(z)$ is a root of the equation

$$\theta = z \exp\left[-2a\left(1 - \sum_{n=1}^{\infty} \pi_n \theta^n\right)\right], \tag{2.8}$$

having the properties (2.6).

By Lagrange's Theorem (Whittaker and Watson [6]), if $|z| < 1$ Eq. (2.8) has only one root θ such that $|\theta| < 1$. From conditions (2.6) this must therefore be the required root $K(z)$ appearing in Eq. (2.7). Making use of Lagrange's expansion an explicit solution will now be derived for $G(\mu, x)$.

We introduce functions $f(\theta)$ and $\varphi(\theta)$, in which z, x and a are regarded as fixed, defined by the following equations:

$$f(\theta) = z^{\delta_{1,\mu}} \exp\left[(\mu x - a)\left(1 - \sum_{n=1}^{\infty} \pi_n \theta^n\right)\right], \tag{2.9}$$

$$\varphi(\theta) = \exp\left[-2a\left(1 - \sum_{n=1}^{\infty} \pi_n \theta^n\right)\right]. \tag{2.10}$$

For $|z| < 1$ the generating function $G(\mu, x)$ is obtained from $f(\theta)$ upon replacing θ by the root $K(z)$ of (2.8) in the interior of the unit circle. Hence by Lagrange's expansion

$$G(\mu, x) = f(0) + \sum_{n=1}^{\infty} \frac{z^n}{n!} \left\{\frac{d^{n-1}}{d\theta^{n-1}} \left[\varphi(\theta)^n \frac{d}{d\theta} f(\theta)\right]\right\}_{\theta=0}. \tag{2.11}$$

From Eq. (2.11) it can be shown directly that the coefficient of z^n in $G(1, x)$, i.e., $P_n(1, x)$, is given by

$$P_1(1, x) = \exp[x - a]$$

$$P_n(1, x) = \frac{a - x}{n - 1} \exp[x - 2na + a] \sum_{j=0}^{n-2} (n - j - 1)\, \gamma_j^{(n-1)} \pi_{n-j-1}, \qquad n \geqslant 2, \tag{2.12}$$

where

$$\gamma_0^{(n)} = 1,$$

$$\gamma_j^{(n)} = \sum_{m=1}^{j} \frac{(2na + a - xm)}{m!} \sum_{i_1 + \cdots + i_m = j} \pi_{i_1} \pi_{i_2} \cdots \pi_{i_m}, \qquad j \geqslant 1, \tag{2.13}$$

the inner summation being over all sets of positive integers $(i_1, i_2, \cdots, i_m)$ such that $i_1 + i_2 + \cdots + i_m = j$.

The probabilities $P_n(-1, x)$ can be found immediately from (2.12) by using the relation

$$P_n(-1, x) = P_{n+1}(1, -x), \qquad n = 0, 1, 2, \cdots, \tag{2.14}$$

which is a direct consequence of Eq. (2.7).

The reflection probabilities $p_n(t)$ defined by (1.1) are obtained from $P_n(-1, x)$ on setting $a = x = t/2$. Thus

$$p_n(t) = \frac{t}{n} \exp\left[-(n+1)\,t\right]$$

$$\times \left[n\pi_n + \sum_{j=1}^{n-1} (n-j)\,\pi_{n-j} \sum_{m=1}^{j} \frac{[(n+1)\,t]^m}{m!} \sum_{i_1+\cdots+i_m=j} \pi_{i_1} \cdots \pi_{i_m}\right], \qquad n \geqslant 2,$$

$$p_0(t) = \exp[-t], \qquad p_1(t) = t\pi_1 \exp[-2t]. \tag{2.15}$$

In the particular case when $\nu = 1$ with probability 1, i.e., if

$$\pi_n = \delta_{1,n}, \qquad n = 1, 2, \cdots, \tag{2.16}$$

then $p_n(t)$ takes the very simple form,

$$p_n(t) = \frac{(n+1)^{n-1}}{n!} t^n \exp\left[-(n+1)\,t\right], \qquad n = 0, 1, 2, \cdots. \tag{2.17}$$

The maximum with respect to t of this function occurs when $t = n/n + 1$ and it will be shown in Section 3 that the critical length for this particular model is $t_c = 1$. This establishes conjecture (1.4) for this model. The proof of conjecture (1.3) follows trivially from equation (2.17).

3. Criticality in the Model of Section 2

The results of Section 2 may be summarized as follows: for $|z| < 1$ the generating functions $G(\mu, x)$ are defined by

$$G(\mu, x) = z^{\delta_{1,\mu}} \exp\left[(\mu x - a)\left(1 - \sum_{n=1}^{\infty} \pi_n K(z)^n\right)\right], \tag{3.1}$$

where $K(z)$ is the unique root of the equation

$$\theta = z \exp\left[-2a\left(1 - \sum_{n=1}^{\infty} \pi_n \theta^n\right)\right], \tag{3.2}$$

in the interior of the unit circle, $|\theta| = 1$. We shall now examine the limiting behavior of $G(\mu, x)$ as z tends to 1 and show that

$$\lim_{z\to 1} G(\mu, x) < 1 \qquad \text{if} \qquad 2a > \frac{1}{\bar{\nu}}, \qquad x \neq \mu a, \tag{3.3}$$

$$\lim_{z\to 1} G(\mu, x) = 1 \qquad \text{otherwise}, \tag{3.4}$$

where $\bar{\nu} = \sum_{n=1}^{\infty} n\pi_n$ is the mean number of additional particle pairs emitted per collision. Thus there is a certain "critical" length of rod, $t_c = 1/\bar{\nu}$, beyond which there is a nonzero probability, $1 - \lim_{z\to 1} G(\mu, x)$, that a single initial particle at x with direction μ will eventually produce an infinite number of descendants. The critical length is usually derived (see Bellman, Kalaba, and Wing [1, 2, 3]) by considering the equation for the mean number of particles produced (derived on the assumption that $\lim_{z\to 1} G(\mu, x) = 1$) and taking as the critical length that length for which the solution first becomes infinite. As we shall see, this procedure leads to the same value, $1/\bar{\nu}$, of t_c. The fact that the mean number of particles produced by a single ancestor is infinite for rod-lengths greater than t_c is clearly implied by (3.3) but not conversely.

To prove (3.3) and (3.4) we first observe that if $x = \mu a$ then $G(\mu, x) = z^{\delta_{1,\mu}}$ so that $\lim_{z\to 1} G(\mu, x) = 1$ for all values of a. It will be assumed in what follows that this trivial case is excluded, i.e., $x \neq \mu a$.

Now if $2a > 1/\bar{\nu}$, then by Lagrange's Theorem, Eq. (3.2) has a unique root $\theta = \psi(z)$ such that $|\psi(z)| < 1$ *for* $|z| \leqslant 1$. By the uniqueness of $K(z)$ and $\psi(z)$ we must have

$$\psi(z) = K(z) \qquad \text{for all } z \text{ such that} \qquad |z| < 1. \tag{3.5}$$

Hence $\lim_{z\to 1} K(z) = \lim_{z\to 1} \psi(z)$ so that

$$|\lim_{z\to 1} K(z)| = |\lim_{z\to 1} \psi(z)| < 1. \tag{3.6}$$

By using this last result, it follows from Eq. (3.1) that

$$\lim_{z\to 1} G(\mu, x) < 1 \qquad \text{if} \qquad 2a > \frac{1}{\bar{\nu}}.$$

This completes the proof of (3.3).

To prove (3.4) we observe from (2.5) that $K(1)$ must be real and less than or equal to 1; $K(1)$ must also satisfy the equation

$$K(1) = \exp\left[-2a\left(1 - \sum_{n=1}^{\infty} \pi_n K(1)^n\right)\right]. \tag{3.7}$$

However if $2a \leqslant 1/\bar{\nu}$ the only real root of this equation which is less than or equal to 1 is $K(1) = 1$. Substituting in Eq. (3.1) gives

$$\lim_{z \to 1} G(\mu, x) = 1 \qquad \text{if} \qquad 2a \leqslant \frac{1}{\bar{\nu}},$$

thus completing the proof of the result (3.4).

Now let us consider the behavior of the mean number, $M(\mu, x)$, of particles ultimately emerging from the right-hand end of the rod conditional upon a single initial particle at x with direction of motion μ, and conditional upon the total number of particles emerging from the right-hand end of the rod being finite. In terms of the generating function $G(\mu, x)$ we have

$$M(\mu, x) = \left\{ [G(\mu, x)]^{-1} \frac{\partial}{\partial z} G(\mu, x) \right\}_{z=1}. \tag{3.8}$$

Hence

$$M(\mu, x) = \delta_{1,\mu} + \frac{(a - x) \sum n\pi_n \gamma^n}{1 - 2a \sum n\pi_n \gamma^n}, \tag{3.9}$$

where $\gamma = \lim_{z \to 1} K(z)$. Setting $\gamma = 1$ in (3.9) we get

$$M(\mu, x) = \delta_{1,\mu} + \frac{(a - x)\,\bar{\nu}}{1 - 2a\bar{\nu}}, \tag{3.10}$$

which is the solution of the equation for the mean,

$$\frac{d}{dx} M(\mu, x) = -\,\mu\bar{\nu}[M(-\,\mu, x) + M(\mu, x)], \tag{3.11}$$

with boundary conditions

$$M(\mu, a\mu) = \delta_{1,\mu}. \tag{3.12}$$

The reason for the discrepancy between (3.9) and (3.10) is that Eq. (3.11) is obtained by differentiating (2.3) with respect to z and setting $z = 1$ and $G(\mu, x) = 1$. It therefore ceases to be valid in the supercritical case when $\lim_{z \to 1} G(\mu, x) < 1$; this is indicated by the fact that when $2a > 1/\bar{\nu}$, $M(\mu, x)$ as given by (3.10) becomes negative (for further discussion see also Moyal [5]).

In the model of Bellman, Kalaba, and Wing only two particles emerge from each collision (one moving in each direction) and the equation corresponding to (3.11) for the mean number of emergent particles (based on the assumption that $\lim_{z \to 1} G(\mu, x) = 1$) is

$$\frac{d}{dx} M(\mu, x) = -\,\mu M(-\,\mu, x), \tag{3.13}$$

with the boundary conditions

$$M(\mu, a\mu) = \delta_{1,\mu}. \tag{3.14}$$

The solution of Eqs. (3.13) and (3.14) can be written

$$M(\mu, x) = \delta_{1,\mu} \frac{\cos(a + x)}{\cos 2a} + \delta_{-1,\mu} \frac{\sin(a + x)}{\cos 2a}. \tag{3.15}$$

Bellman, Kalaba, and Wing took the critical length for their model to be the smallest rod-length for which this solution becomes infinite. Thus from (3.15) we find that the critical length, t_c, is equal to $\pi/2$. Applying the same argument to the model considered previously, we find from (3.10) that the critical length is $t_c = 1/\bar{\nu}$. Thus we arrive at the same value as was previously derived by considering the behavior of $\lim_{z\to 1} G(\mu, x)$.

4. Criticality in the Model of Bellman, Kalaba, and Wing

It has been shown that for the model of Section 2 there is a critical length $t_c = 1/\bar{\nu}$ such that

$$\lim_{z\to 1} G(\mu, x) < 1 \qquad \text{if} \qquad 2a > t_c, \qquad x \neq \mu a, \tag{4.1}$$

$$\lim_{z\to 1} G(\mu, x) = 1 \qquad \text{otherwise.} \tag{4.2}$$

The same kind of behavior will now be established for the model of Bellman, Kalaba, and Wing in which, at each collision, the incident particle is lost and two new particles emerge (one in each direction) from the point of collision.

Instead of considering, as in Section 2, the probability $P_n(\mu, x)$ that a single particle originating at x $(-a \leqslant x \leqslant a)$ with direction μ ultimately generates n particles emerging from the right-hand end of the rod, it is more convenient now to consider the probability $Q_n(\mu, x)$ that the *total* number of particles ultimately emerging from the rod is n. The corresponding generating function is defined by

$$H(\mu, x) = \sum_{n=0}^{\infty} Q_n(\mu, x)\, z^n, \qquad |z| \leqslant 1.$$

Clearly

$$\sum_{n=0}^{\infty} P_n(\mu, x) = \sum_{n=0}^{\infty} Q_n(\mu, x) = \lim_{z\to 1} H(\mu, x)$$

so that it will be sufficient to prove (4.1) and (4.2) with G replaced by H. In fact we shall prove the following:

$$\lim_{z\to 1} H(\mu, x) < 1 \qquad \text{if} \qquad 2a > \frac{\pi}{2}, \qquad x \neq \mu a, \tag{4.3}$$

$$\lim_{z\to 1} H(\mu, x) = 1 \qquad \text{otherwise.} \tag{4.4}$$

The critical length t_c is thus $\pi/2$ as already found by Bellman, Kalaba, and Wing from their study of the mean number of particles produced by a single ancestor (see Section 3).

Proceeding as in Section 2 we find that the equations for the generating functions $H(\mu, x)$ are

$$\frac{d}{dx} H(\mu, x) = \mu H(\mu, x)\,[1 - H(-\mu, x)], \tag{4.5}$$

$$H(\mu, \mu a) = z. \tag{4.6}$$

Introducing new functions $u(x, z)$ and $v(x, z)$ defined by

$$u(x, z) = \tfrac{1}{2}\,[H(1, x) + H(-1, x)]\,, \tag{4.7}$$

$$v(x, z) = \tfrac{1}{2}\,[H(1, x) - H(-1, x)]\,, \tag{4.8}$$

we find from (4.5) that

$$\frac{du}{dx} = v, \tag{4.9}$$

$$\frac{dv}{dx} = u - u^2 + v^2. \tag{4.10}$$

Inspection of Eq. (4.5) shows that

$$\log H(1, x) - H(1, x) + \log H(-1, x) - H(-1, x) = \chi(z), \tag{4.11}$$

where $\chi(z)$ is independent of x.

Suppose that $H(1, 0) = \alpha(z)$. Then by symmetry $H(-1, 0)$ is also equal to $\alpha(z)$. Since a particle which starts at $x = a$ moving to the right can have no collisions, we must have $H(1, a) = z$. We denote by $\beta(z)$ the corresponding function $H(-1, a)$. For real positive z less than or equal to 1, Eq. (4.5) shows that

$$\beta(z) \leqslant H(-1, x) \leqslant \alpha(z) \leqslant H(1, x) \leqslant z \qquad \text{for all} \qquad x \geqslant 0. \tag{4.12}$$

Thus in order to prove (4.3), it will be sufficient (by using (4.5) and (4.12)) to show that

$$\lim_{z\to 1} \alpha(z) < 1 \qquad \text{for} \qquad 2a > \frac{\pi}{2}. \tag{4.13}$$

To prove (4.4) it will be sufficient (by using (4.5) and (4.12)) to show that

$$\lim_{z\to 1} \alpha(z) = 1 \qquad \text{for} \qquad 2a \leqslant \frac{\pi}{2}. \tag{4.14}$$

Substituting the values of $H(1, x)$ and $H(-1, x)$ at $x = 0$ and $x = a$ in equation (4.11) we find that

$$2[\log \alpha(z) - \alpha(z)] = \log [z\beta(z)] - \beta(z) - z = \chi(z). \tag{4.15}$$

Rewriting Eq. (4.11) in terms of $u(x, z)$ and $v(x, z)$ and using the relation between $\alpha(z)$ and $\chi(z)$ gives

$$u^2 - v^2 = \alpha^2 e^{-2(\alpha - u)}, \tag{4.16}$$

and hence from (4.9)

$$\frac{du}{dx} = \sqrt{u^2 - \alpha^2 e^{-2(\alpha-u)}}, \tag{4.17}$$

the positive value of the square root being appropriate if z is real, positive and less than or equal to 1, and if $x \geqslant 0$.

Integrating Eq. (4.17) between the limits $x = 0$ and $x = a$ leads to the relationship

$$a = \int_{\alpha(z)}^{\frac{1}{2}[z+\beta(z)]} \frac{dy}{\sqrt{y^2 - \alpha^2 e^{2(y-\alpha)}}}, \; 0 < z < 1, \tag{4.18}$$

where

$$z\beta e^{-(z+\beta)} = \alpha^2 e^{-2\alpha}.$$

We shall now use Eq. (4.18) to deduce the limiting behavior of $\alpha(z)$ as z tends to 1.

Suppose that z is real and positive and $z \leqslant 1$ so that $\alpha(z)$ and $\beta(z)$ are both real and positive and $\alpha \leqslant 1$, $\beta \leqslant 1$. Define $\gamma(z)$ to be unique real root of the equation

$$\gamma e^{-(1+\gamma)} = \alpha^2 e^{-2\alpha}$$

such that $\gamma \leqslant 1$ for $z \leqslant 1$. It then follows from the relations

$$\beta z e^{-(z+\beta)} = \alpha^{-2\alpha} \qquad \text{and} \qquad \beta(z) \leqslant z$$

that $z + \beta(z) \leqslant 1 + \gamma(z)$.

We wish to show that $\lim_{z\to 1} \alpha(z) < 1$ if $2a > \pi/2$ and that $\lim_{z\to 1} \alpha(z) = 1$ if $2a \leqslant \pi/2$. Suppose first that $\lim_{z\to 1} \alpha(z) = 1$. Then from Eq. (4.18)

$$
\begin{aligned}
a &= \lim_{z\to 1} \int_{\alpha(z)}^{\frac{1}{2}[z+\beta(z)]} \frac{dy}{\sqrt{y^2 - \alpha^2 e^{2(y-\alpha)}}} \\
&\leqslant \lim_{z\to 1} \int_{\alpha(z)}^{\frac{1}{2}[1+\gamma(z)]} \frac{dy}{\sqrt{y^2 - \alpha^2 e^{2(y-\alpha)}}} \\
&= \lim_{\delta\to 0} \int_{1-\delta}^{1-\delta/\sqrt{2}} \frac{dy}{\sqrt{y^2 - (1-\delta)^2\, e^{2(y-1+\delta)}}} \\
&= \lim_{\delta\to 0} \int_{\delta/\sqrt{2}}^{\delta} \frac{dv}{\sqrt{\delta^2 - v^2}} = \frac{\pi}{4}.
\end{aligned}
$$

Thus if $\lim_{z\to 1} \alpha(z) = 1$ then $a \leqslant \pi/4$, i.e., if $2a > \pi/2$ then $\lim_{z\to 1} \alpha(z) < 1$. Conversely if $\lim_{z\to 1} \alpha(z) = \alpha(1) < 1$ then

$$
a = \int_{\alpha(1)}^{\frac{1}{2}[1+\beta(1)]} \frac{dy}{\sqrt{y^2 - \alpha(1)^2\, e^{2[y-\alpha(1)]}}} > \frac{\pi}{4},
$$

and hence $\lim_{z\to 1} \alpha(z) = 1$ if $2a \leqslant \pi/2$. This completes the proof of the properties (4.1) and (4.2) for the model of Bellman, Kalaba, and Wing.

5. A Model which Includes Scattering and Absorption

Finally we consider a model which is a generalization of a model previously considered by Harris [4]. We suppose that as a result of each collision ν particles are emitted where ν is a random variable taking the values $0, 1, 2, \cdots$, with probabilities $\pi_0, \pi_1, \pi_2, \cdots$. Each particle emerging after a collision moves to the right or left with probability $\frac{1}{2}$ independently of the other particles. We observe that π_0 is the probability that the colliding particle is absorbed, π_1 is the probability that it is scattered and $\sum_{n=2}^{\infty} \pi_n$ is the probability that multiplication occurs. It is also of interest to notice that the process can be considered as the projection on the x-axis of the corresponding process in the three-dimensional slab $\{\mathbf{x} : -a \leqslant x \leqslant a\}$ when scattering is isotropic and the projected distances between collisions are assumed to be independently distributed with a negative exponential distribution. Such an approximation has been used by Fermi [7] and improved by Brockwell [8] in the case where there is no multiplication of particles.

Defining the generating functions $H(\mu, x)$, $u(x, z)$ and $v(x, z)$ exactly as in Section 4, we find that they satisfy the equations

$$\mu \frac{d}{dx} H(\mu, x) = H(\mu, x) - \sum_{n=0}^{\infty} 2^{-n}\pi_n[H(\mu, x) + H(-\mu, x)]^n, \tag{5.1}$$

$$H(\mu, \mu a) = z, \tag{5.2}$$

or, in terms of u and v,

$$\frac{d}{dx} u(x, z) = v(x, z), \tag{5.3}$$

$$\frac{d}{dx} v(x, z) = u(x, z) - \sum_{n=0}^{\infty} \pi_n u(x, z)^n, \tag{5.4}$$

with boundary conditions

$$u(a, z) + v(a, z) = z, \tag{5.5}$$

$$v(0, z) = 0. \tag{5.6}$$

We shall suppose throughout that the distribution defined by the probabilities π_n has a mean, $m = \sum_{n=0}^{\infty} n\pi_n > 1$, and that z is real, positive and less than or equal to 1. If we consider a particle which starts at a point x and moves and multiplies on the real line $(-\infty, \infty)$ in the manner described, then a well-known result in the theory of branching processes gives the probability that it ultimately produces only a finite number of descendants as the unique real root, p, in $[0, 1)$ of the equation

$$p = \sum_{n=0}^{\infty} \pi_n p^n. \tag{5.7}$$

For the process in the finite rod $[-a, a]$ it follows that $u(x, 1) > p$, $-a \leqslant x \leqslant a$. Thus there exists $\epsilon (0 < \epsilon \leqslant 1)$ such that $p < u(0, z) < 1$ provided $1 - \epsilon < z < 1$. Noting that

$$y - \sum_{n=0}^{\infty} \pi_n y^n > 0 \qquad \text{if} \qquad p < y < 1, \tag{5.8}$$

it can easily be shown from (5.3) and (5.4) that for all x in $(0, a]$

$$v(x, z) > 0 \qquad \text{provided} \qquad 1 - \epsilon < z < 1. \tag{5.9}$$

We shall assume in all that follows that z belongs to the interval $(1 - \epsilon, 1)$.

Then, writing $u(0, z) = \tau(z)$ and $u(a, z) = \rho(z)$, it follows from (5.3) and (5.4) that

$$v(x, z) = \left\{u(x, z)^2 - \tau(z)^2 - 2 \sum_{n=0}^{\infty} \frac{\pi_n}{n+1} [u(x, z)^{n+1} - \tau(z)^{n+1}]\right\}^{1/2}, \quad (5.10)$$

$$x = \int_{\tau(z)}^{u(x,z)} \left\{y^2 - \tau(z)^2 - 2 \sum_{n=0}^{\infty} \frac{\pi_n}{n+1} [y^{n+1} - \tau(z)^{n+1}]\right\}^{-1/2} dy, \quad (5.11)$$

the positive value of the square root being appropriate for $x \geqslant 0$. The unknown functions $\tau(z)$ and $\rho(z)$ are uniquely determined (in the range $1 - \epsilon < z < 1$) by the boundary conditions (5.5) and (5.6) which can be written

$$z = \rho(z) + \left\{\rho(z)^2 - \tau(z)^2 - 2 \sum_{n=0}^{\infty} \frac{\pi_n}{n+1} [\rho(z)^{n+1} - \tau(z)^{n+1}]\right\}^{1/2}, \quad (5.12)$$

$$a = \int_{\tau(z)}^{\rho(z)} \left\{y^2 - \tau(z)^2 - 2 \sum_{n=0}^{\infty} \frac{\pi_n}{n+1} [y^{n+1} - \tau(z)^{n+1}]\right\}^{-1/2} dy. \quad (5.13)$$

Equations (5.12) and (5.13) correspond to the equations (4.18) which were derived in Section 4 for the model of Bellman, Kalaba, and Wing. Notice that if $\pi_0 = 0$ then Eqs. (5.10)-(5.13) are valid for all real z in $0 < z < 1$.

The arguments of Section 4 can be applied to Eqs. (5.12) and (5.13) to deduce the limiting behavior of $\tau(z)$ and $H(\mu, x)$ as z tends to 1. Proceeding in this way (recalling that $m > 1$) it is found that

$$\lim_{z \to 1} H(\mu, x) < 1 \qquad \text{if} \qquad 2a > \frac{2}{\sqrt{m-1}} \arcsin m^{-1/2}, \qquad x \neq \mu a, \quad (5.14)$$

$$\lim_{z \to 1} H(\mu, x) = 1 \qquad \text{otherwise.} \quad (5.15)$$

(If $m \leqslant 1$ then $\lim_{z\to 1} H(\mu, x) = 1$ for all values of a.)

Furthermore we can use Eqs. (5.12) and (5.13) to determine $\lim_{z\to 1} H(\mu, x)$ numerically for rods of any length. Assuming a value of $\tau(1)$ (such that $p < \tau(1) < 1$), the corresponding value of $\rho(1)$ is determined by (5.12) and the corresponding rodlength $2a$ is then found from (5.13) by numerical integration. Knowing the value of $\tau(1)$ corresponding to a given rod-length, we can then use (5.10) and (5.11) to determine $u(x, 1)$, $v(x, 1)$ and hence $\lim_{z\to 1} H(\mu, x)$ for a rod of this length. We observe in particular that as the rod-length, $2a$, tends to infinity, $\tau(1)$ tends to p and $\rho(1)$ tends to the unique root ρ (such that $p < \rho < 1$) of the equation

$$1 = \rho + \left\{\rho^2 - p^2 - 2 \sum_{n=0}^{\infty} \frac{\pi_n}{n+1} [\rho^{n+1} - p^{n+1}]\right\}^{1/2}. \quad (5.16)$$

Figure 1 shows the graph of $\lim_{z\to 1} H(-1, a)$ determined as described above for the particular case in which exactly two particles are emitted at each collision. This is the model considered by Harris [4]. The corresponding function for the model of Section 4 is also shown. This was derived by applying exactly the same technique to Eqs. (4.16), (4.17), and (4.18).

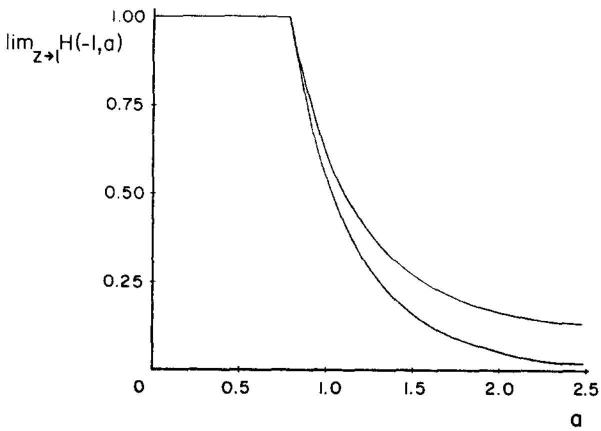

FIG. 1. The probability that a single particle incident at one end of a rod of length 2a produces a finite number of emergent particles. The upper curve, which is asymptotic to the value 0.11575, refers to the model of Harris; the lower curve, which is asymptotic to zero, refers to the model of Bellman, Kalaba, and Wing.

Let us denote by $1 - \delta$ the value of $\tau(1)$ for a rod whose length, $2a$, is slightly greater than the critical length, $2/\sqrt{m-1}$ arcsin $m^{-1/2}$. If the distribution defined by the probabilities π_n possesses a second moment $s = \sum_{n=0}^{\infty} n^2\pi_n$, we can use Eq. (5.12) to derive the asymptotic expression

$$\rho(1) = 1 + k_1\delta + k_2\delta^2 + o(\delta^2),$$

where

$$k_1 = -\sqrt{1 - m^{-1}} \qquad \text{and} \qquad k_2 = \frac{s - m}{6\sqrt{m^2 - m}}\,[1 - (1 - m^{-1})^{3/2}],$$

and hence, from (5.13),

$$a = \frac{1}{\sqrt{m-1}} \arcsin m^{-1/2} + \delta\,\frac{(s-m)(3m-1)}{6\sqrt{m^3(m-1)^3}} + o(\delta). \tag{5.17}$$

This generalizes a result which was obtained by Harris [4] for the particular case in which exactly two particles are emitted at each collision.

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References

1. R. Bellman, R. Kalaba, and G. M. Wing. On the principle of invariant imbedding and one-dimensional neutron multiplication. *Proc. Natl. Acad. Sci. USA* **43** (1957), 517-520.
2. R. Bellman, R. Kalaba, and G. M. Wing. On the principle of invariant imbedding and neutron transport theory; I—One-dimensional case. *J. Math. Mech.* **7** (1958), 149-162.
3. R. Bellman and R. Kalaba. Transport theory and invariant imbedding. *Proc. Symp. Appl. Math.* **11** (1961), 206-218.
4. T. E. Harris. On one-dimensional neutron multiplication, The Rand Corporation, RM-2317 (1960).
5. J. E. Moyal. A general theory of first-passage distributions in transport and multiplicative processes. To appear in *J. Math. Phys.*
6. E. T. Whittaker and G. N. Watson. "A Course of Modern Analysis," 4th ed., Cambridge Univ. Press, Cambridge, 1952.
7. E. Fermi. On the motion of neutrons in hydrogenous substances. "Collected Papers of Enrico Fermi," Vol. 1, pp. 980-1016. University of Chicago, Chicago, 1962.
8. P. J. Brockwell. Generalized one-dimensional scattering processes with applications in three dimensions. *Nuovo Cimento, Ser. 10,* **36** (1964), 370-384.
9. T. W. Mullikin and R. N. Snow. One-dimensional neutron processes, The Rand Corporation, D-6849 (1959).